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STRAIN-GRADIENT EFFECTS AROUND SPHERICAL INCLUSIONS AND CAVITIES

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Abstract—This paper deals with the elastic stress field due to a spherical inclusion or cavity in an infinite exterior region subjected to spherically symmetric tension at infinity. The results are derived for a potential energy that depends on strains and strain gradients. Such a potential energy will give rise to both classical stresses and non-classical hyperstresses. It is shown that the hyperstresses are confined to the vicinity of the interface or free surface associated with an inclusion or a cavity and are therefore representative of skin effects. The stress concentration factor, computed on the basis of the hoop stresses around a cavity, is shown to be at large variance with the classical value.

NOTATION

Symbol	Equivalent Symbol in [2], Sections 12, 13	
$\bar{c}$	$\bar{a}_1$	micro elasticity modulus
c	$\bar{a}_2$	micro elasticity modulus
f	$\bar{f}$	micro elasticity modulus
l	l_1	characteristic measure of strain gradient theory
r, θ, ϕ		spherical coordinates
$\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$		unit vectors in spherical coordinates
r_0		radius of inclusion or cavity
$\mathbf{P}$	$\mathbf{P}$	boundary traction
$\mathbf{Q}$	$\mathbf{Q}$	non-self-equilibrating double traction on boundary
R	R	self-equilibrating double traction on boundary
P_r		radial component of $\mathbf{P}$ on $r = r_0$
$P_{\theta/\theta}$		tangential component of $\mathbf{P}$ on surface $\theta = \theta_0$ at $r = r_0$
$P_{\phi/\phi}$		circumferential component of $\mathbf{P}$ on surface $\phi = \phi_0$ at $r = r_0$
B_0	B_0	stress function
$\mathbf{u}$	$\mathbf{u}$	displacement vector
α		positive dimensionless quantity
$\rho = r_0/l$		dimensionless characteristic measure
λ, μ	$\bar{\lambda}, \bar{\mu}$	Lamé moduli in presence of strain-gradient effects
ε_r		radial strain
σ_θ		tangential stress in spherical coordinates
T		magnitude of hydrostatic tension in psi, selected to be unity
$\mathfrak{K}, \hat{\mathfrak{K}}$	$\mathfrak{K}, \hat{\mathfrak{K}}$	kinematical quantities in strain-gradient theory
$\bar{\mu}_{\theta\phi}, \bar{\mu}_{r\theta\theta}, \bar{\mu}_{rrr}$		hyperstresses

Subscripts (or superscripts) 1 and 2 refer to the inclusion and exterior respectively.

INTRODUCTION

IN THIS paper it is assumed that the potential energy of an elastic body is a function of both strain and strain gradient. The boundary conditions and stress-equations of equilibrium for such a body were first derived by Toupin [1]. The constitutive equations for the linear, isotropic, centrosymmetric case were given by Mindlin [2] in three alternative forms. Mindlin has shown that these various forms lead to a unique system of displacement equations of equilibrium for which he presented a complete solution in terms of stress functions.

An elastic potential energy which depends on strains and strain gradients will give rise to classical "Cauchy Stresses" and higher order stress-terms ("Hyperstresses"). These hyperstresses are couple-stresses and double-stresses, some of which are depicted in Fig. 1.

For an isotropic, centrosymmetric material the constitutive relations contain five additional elastic constants (micromoduli) whose dimensions differ from those of the Lamé constants by the square of length. It is possible to relate certain algebraic combinations of these micromoduli to the Lamé moduli by means of two characteristic length parameters.

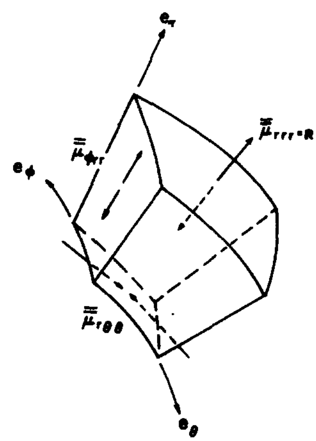

FIG. 1. Some hyperstresses in spherical coordinates.

As already observed on previous occasions [3, 4], the presence of higher-order kinematical terms (e.g. strain gradients) in the potential energy function introduces a higher order contact at the interface between dissimilar materials. The higher order contact conditions may require the materials to interact across their mutual interface so as to form a common boundary layer, rather than only an infinitesimally thin common surface as in the classical case.

The solution to an elasticity problem based upon the strain-gradient theory can be expressed in three different ways, which depend on the specific form selected for the potential energy. These forms have been presented by Mindlin ([2], sections 9, 11 and 12). For convenience it has been decided to follow the third form of Mindlin's equation ([2], section 12) because it will lead to a solution that contains only two of the five micromoduli and involve only one characteristic length. The other forms lead to solutions that depend on two combinations of all five micromoduli. The reader is referred to [2], especially sections 9–13 for a detailed exposition of the strain-gradient theory.

ANALYSIS

Consider a spherical inclusion (or cavity) of radius $r = r_0$ in an infinite elastic exterior region which is subjected to hydrostatic, spherically symmetric tension as shown in Fig. 2.

Assume that the potential energy of the inclusion and the exterior region depends on strains and strain-gradients. Denote the elastic constants of the inclusion by $\lambda_1, \mu_1, l_1, c_1, \bar{c}_1$

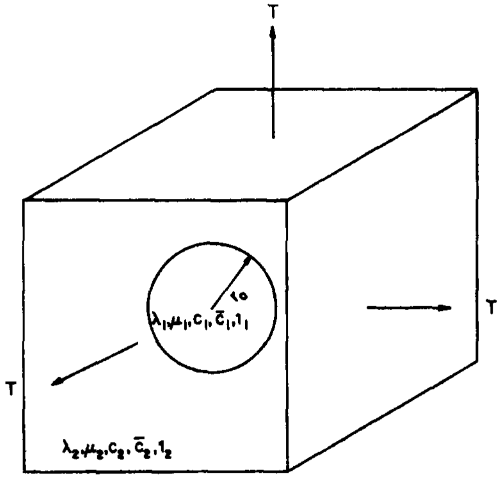

FIG. 2. A spherical inclusion within an infinite region subjected to hydrostatic tension.

and those of the exterior by $\lambda_2, \mu_2, l_2, c_2, \bar{c}_2$.* The relation between these quantities and their counterparts in [2], section 12, is given in the notation. Let the spherical coordinates be r, θ, ϕ with unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$. Due to the complete spherical symmetry the displacement vector is given by

$$\mathbf{u} = u_r(r)\mathbf{e}_r. \tag{1}$$

In the following we shall write

$$u_r(r) = u.$$

The non-vanishing components of strain are

$$\varepsilon_r = u', \qquad \varepsilon_\theta = \varepsilon_\phi = \frac{u}{r}. \tag{2}$$

It is found that all rotation gradients vanish so that, following [2], equation 12.2, we have

$$\bar{\bar{\kappa}}_{ijk} = \hat{\kappa}_{ijk}$$

which yield the following non-vanishing components

$$\hat{\kappa}_{rrr} = u''$$

$$\hat{\kappa}_{r\theta\theta} = \hat{\kappa}_{r\phi\phi} = \hat{\kappa}_{\theta r\theta} = \hat{\kappa}_{\theta\theta r} = \hat{\kappa}_{\phi r\phi} = \hat{\kappa}_{\phi\phi r} = \frac{u'}{r} - \frac{u}{r^2} \tag{3}$$

where primes denote derivatives with respect to r. Equations (12.7) of [2] yield the following non-vanishing components of stresses and hyperstresses

$$\sigma_r = (\lambda + 2\mu)u' + 2\lambda\frac{u}{r}$$

* Each material possesses, in addition, three micromoduli and a characteristic length which are left unnamed, since they are irrelevant for what follows.

$$\sigma_\theta = \sigma_\phi = \lambda u' + 2(\lambda+\mu)\frac{u}{r}$$

$$\bar{\mu}_{\theta\phi} = -\bar{\mu}_{\phi\theta} = f\left(u'' + \frac{2}{r}u' - \frac{2}{r^2}u\right)$$

$$\bar{\bar{\mu}}_{rrr} = 3\bar{c}\left(u'' + \frac{2}{r}u' - \frac{2}{r^2}u\right) + 2cu''$$

$$\bar{\bar{\mu}}_{r\theta\theta} = \bar{\bar{\mu}}_{r\phi\phi} = \bar{\bar{\mu}}_{\theta r\theta} = \bar{\bar{\mu}}_{\theta\theta r} = \bar{\bar{\mu}}_{\phi r\phi} =$$

$$\bar{\bar{\mu}}_{\phi\phi r} = \bar{c}\left(u'' + \frac{2}{r}u' - \frac{2}{r^2}u\right) + 2c\left(\frac{u'}{r} - \frac{u}{r^2}\right). \tag{4}$$

Following [2], equations (12.14), we express the boundary tractions and hypertractions, on the surface $r = r_0$, in terms of u

$$P_r = (\lambda+2\mu)u' + 2\lambda\frac{u}{r} - 3\bar{c}\left(u''' + \frac{2}{r}u'' - \frac{4}{r^2}u' + \frac{4}{r^3}u\right)$$

$$-2c\left(u''' + \frac{2}{r}u'' - \frac{6}{r^2}u' + \frac{6}{r^3}u\right) \tag{5}$$

$$R = \bar{\bar{\mu}}_{rrr} \tag{6}$$

$$P_\theta = P_\phi = \mathbf{Q} = 0. \tag{7}$$

Expressions (5), (6) and (7) can be obtained by employing Mindlin's forms I and II ([2], sections 9 and 11).

The hoop double-tractions $\bar{\bar{\mu}}_{r\theta\theta}$ and $\bar{\bar{\mu}}_{r\phi\phi}$ do however depend on the form utilized since their meaning is distinct in each case. The quantities $P_{\theta/\theta}$ and $P_{\phi/\phi}$, which are the counterparts of the traction P_r, act on the surfaces $\theta = \theta_0$ and $\phi = \phi_0$ in the directions $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ respectively and also depend on the specific form being employed. These quantities represent some kind of resultants of long range and short range forces which may be of interest in studies of stress concentrations since it is not yet obvious how one might determine the crucial factor underlying fracture or yield within the scope of strain gradient theory. Employing form III, we obtain

$$P_{\theta/\theta} = P_{\phi/\phi} = \lambda u' + 2(\lambda+u)\frac{u}{r} - 3\bar{c}\left(u''' + \frac{4}{r}u''\right) - 6c\frac{u''}{r}. \tag{8}$$

The solution to our problem can be expressed in terms of uniform triaxial tension $[U_3]$ and stress functions B_0 which satisfy the equation

$$(1 - l^2\nabla^2)\nabla^2 B_0 = 0 \tag{9}$$

The functions B_0 which participate in the solution to the present problem are

$$B_{01} = 2\frac{\lambda+2\mu}{\mu}\frac{r_0^3}{r}\sinh\frac{r}{l}$$

$$B_{02} = \frac{\lambda+2\mu}{\lambda+\mu}\frac{r_0^3}{r} \tag{10}$$

$$B_{03} = 2\frac{\lambda+2\mu}{\mu}\frac{r_0^3}{r}e^{-r/l}.$$

For a spherically symmetric field the radial displacement u is derived from B_0 as follows

$$u = \frac{l^2}{2}\left(B_0''' + \frac{2}{r}B_0'' - \frac{2}{r^2}B_0'\right) - \frac{1}{2}\frac{\lambda+\mu}{\lambda+2\mu}B_0'. \tag{11}$$

Inserting (11) into (4), (5) and (6) and denoting $\xi = r/l$, $\rho = r_0/l$ one obtains the following fields: For B_{01}:

$$u = \frac{r_0^3}{r^2}(-\sinh\xi + \xi\cosh\xi)$$

$$\varepsilon_r = \frac{r_0^3}{r^3}(2\sinh\xi - 2\xi\cosh\xi + \xi^2\sinh\xi)$$

$$\sigma_\theta = \frac{r_0^3}{r^3}[\lambda\xi^2\sinh\xi + 2\mu(-\sinh\xi + \xi\cosh\xi)]$$

$$\bar{\bar{\mu}}_{r\theta\theta} = \frac{r_0^3}{r^2}\left[\frac{\bar{c}}{l^2}(-\sinh\xi + \xi\cosh\xi) + 2\frac{c}{l^2}(3\xi^{-2}\sinh\xi - 3\xi^{-1}\cosh\xi + \sinh\xi)\right]$$

and, at $r = r_0$ (12)

$$P_r = \lambda\rho^2\sinh\rho + 2\mu(2\sinh\rho - 2\rho\cosh\rho + \rho^2\sinh\rho) - 3\frac{\bar{c}}{l^2}(\sinh\rho - 2\rho\cosh\rho + \rho^2\sinh\rho) - 2\frac{c}{l^2}(-6\rho^{-2}\sinh\rho + 6\rho^{-1}\cosh\rho - 2\rho\cosh\rho + \rho^2\sinh\rho)$$

$$R = 3\frac{\bar{c}}{l^2}(-\sinh\rho + \rho\cosh\rho) + 2\frac{c}{l^2}(-6\rho^{-2}\sinh\rho + 6\rho^{-1}\cosh\rho - 3\sinh\rho + \rho\cosh\rho).$$

For B_{02}:

$$u = \frac{1}{2}\frac{r_0^3}{r^2}$$

$$\varepsilon_r = -\frac{r_0^3}{r^3}, \qquad \sigma_\theta = \frac{r_0^3}{r^3}\mu, \qquad \bar{\bar{\mu}}_{r\theta\theta} = -3\frac{r_0^3}{r^2}\frac{c}{l^2}\xi^{-2}$$

at $r = r_0$, (13)

$$P_r = -2\left(\mu + 3\frac{c}{l^2}\rho^{-2}\right), \qquad R = 6\frac{c}{l^2}\rho^{-2}.$$

For B_{03}:

$$u = -e^{-\xi}\frac{r_0^3}{r^2}(1+\xi), \qquad \varepsilon_r = e^{-\xi}\frac{r_0^3}{r^3}(2+2\xi+\xi^2),$$

$$\sigma_\theta = -e^{-\xi}\frac{r_0^3}{r^3}[-\lambda\xi^2 + 2\mu(1+\xi)]$$

$$\bar{\bar{\mu}}_{r\theta\theta} = -e^{-\xi}\frac{r_0^3}{r^2}\left[\frac{\bar{c}}{l^2}(1+\xi) - 2\frac{c}{l^2}(3\xi^{-2}+3\xi^{-1}+1)\right]$$

and, at $r = r_0$ (14)

$$P_r = -e^{-\rho}\left[-\lambda\rho^2 - 2\mu(2+2\rho+\rho^2) + 3\frac{\bar{c}}{l^2}(2+2\rho+\rho^2) - 2\frac{c}{l^2}(6\rho^{-2}+6\rho^{-1}-2\rho-\rho^2)\right]$$

$$R = -e^{-\rho}\left[3\frac{\bar{c}}{l^2}(1+\rho) + 2\frac{c}{l^2}(6\rho^{-2}+6\rho^{-1}+3+\rho)\right].$$

The uniform triaxial field $[U_3]$ yields

$$u = \frac{1}{3\lambda+2\mu}r, \qquad \varepsilon_r = \frac{1}{3\lambda+2\mu}, \qquad \sigma_\theta = 1$$

$$\bar{\bar{\mu}}_{r\theta\theta} = 0, \qquad P_r = 1, \qquad R = 0. \tag{15}$$

Equations (12)–(14) contain the quantities l, c and $\bar{c}$, only two of which are independent since they are interrelated through the expression ([2], equation $(12.16)_1$).

$$l^2 = \frac{3\bar{c}+2c}{\lambda+2\mu}. \tag{16}$$

It is advantageous to retain l as an independent parameter since it appears in the exponents of equations (12)–(14). We therefore introduce a dimensionless parameter α and write

$$c = \alpha(\lambda+2\mu)l^2$$

thus (17)

$$\bar{c} = \frac{1-2\alpha}{3}(\lambda+2\mu)l^2.$$

The quantity α determines the ratio between c and $\bar{c}$ and so it influences the relations between the double stresses and the strain-gradients.

Utilizing (17) we now rewrite (4), (5), (8), (12), (13) and (14) in terms of α and l and express the solution by means of these two independent parameters. Since the nature of the substitution (17) is strictly algebraic it is possible to restate any result in terms of c and $\bar{c}$ by resubstitution. Such a need may arise in the reduction to classical theory. This reduction might necessitate an examination of various iterated limits as c and $\bar{c}$ approach zero. A study of such limits of results expressed in terms of α and l by allowing $l \to 0$ is restricted to the case of c and $\bar{c}$ approaching zero simultaneously.

The solution to the problem consists of two parts, $[S_1]$ and $[S_2]$, which apply to the inclusion and the exterior region, respectively. For the inclusion

$$[S_1] = A[U_3]+B[B_{01}] \qquad (0 \leqslant r \leqslant r_0). \tag{18}$$

For the exterior

$$[S_2] = [U_3]+C[B_{02}]+D[B_{03}] \qquad (r_0 \leqslant r < \infty). \tag{19}$$

There are various admissible sets of boundary conditions at the interface $r = r_0$ [3], [4], [5]:

(a) *Higher order contact and complete transmissibility of energy*

In this case the boundary conditions at $r = r_0$ are

$$\begin{aligned} u^{(1)} &= u^{(2)} \qquad & \varepsilon_r^{(1)} &= \varepsilon_r^{(2)} \\ P_r^{(1)} &= P_r^{(2)} \qquad & R^{(1)} &= R^{(2)}. \end{aligned} \tag{20}$$

Superscripts (1) and (2) refer to the inclusion and exterior region respectively.

Conditions (20) yield four equations in the four unknowns A, B, C and D. Upon solving for these coefficients one obtains the following expressions at $r = r_0$:

$$\begin{aligned} \sigma_\theta^{(1)} = \tfrac{3}{8}\mu_1\mu_2(\lambda_2+\mu_2)\frac{e^{-\rho_2}}{\Delta}\Bigg\{ & -\frac{NV}{\lambda_2+2\mu_2}\cdot\frac{1+\rho_2}{\lambda_1+2\mu_1} \\ & +\frac{V}{U}\left[\frac{1+\rho_2}{\lambda_1+2\mu_1}+(3+3\rho_2+\rho_2^2)\frac{W}{\lambda_2+2\mu_2}\right] \\ & -\frac{1}{(\lambda_1+2\mu_1)(\lambda_2+2\mu_2)}\;\frac{1+\rho_2}{3\lambda_1+2\mu_1}\,[\lambda_1\rho_1^2\sinh\rho_1-2\mu_1(\sinh\rho_1-\rho_1\cosh\rho_1)]\Bigg\} \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{1}{r_0}\bar{\bar{\mu}}_{r\theta\theta}^{(1)} = & -\tfrac{1}{8}\mu_1\mu_2\frac{e^{-\rho_2}}{\Delta}\,\frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2}\,\frac{1+\rho_2}{3\lambda_1+2\mu_1}\,[-\sinh\rho_1+\rho_1\cosh\rho_1 \\ & +2\alpha_1(9\rho_1^{-2}\sinh\rho_1-9\rho_1^{-1}\cosh\rho_1+4\sinh\rho_1-\rho_1\cosh\rho_1] \end{aligned} \tag{22}$$

$$\begin{aligned} \sigma_\theta^{(2)} = \tfrac{1}{8}\mu_1\mu_2\,e^{-\rho_2}\,\frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2}\,\frac{1}{3\lambda_1+2\mu_1}\,\frac{V}{\Delta}\Bigg\{ & \frac{3\lambda_2+2\mu_2}{\lambda_2+2\mu_2}\,\mathrm{W}\Bigg[3(1+\rho_2) \\ & +\frac{(3\lambda_1+2\mu_1+4\mu_2)(3+3\rho_2+\rho_2^2)}{U}+\rho_2^2\Bigg] \\ & +\left[2\mu_2+\frac{(3\lambda_2+2\mu_2)(3\lambda_1+2\mu_1+4\mu_2)}{U}\right]\frac{1+\rho_2}{\lambda_1+2\mu_1}\Bigg\} \end{aligned} \tag{23}$$

$$\frac{1}{r_0}\bar{\mu}^{(2)}_{r\theta\theta} = -\tfrac{1}{4}\mu_1\mu_2\,\mathrm{e}^{-\rho_2}\frac{\lambda_2+\mu_2}{3\lambda_1+2\mu_1}\frac{V}{\Delta}\left\{3\alpha_2\rho_2^{-2}\left[\frac{1+\rho_2}{\lambda_1+2\mu_1}\right.\right.$$
$$\left.\left.+(3+3\rho_2+\rho_2^2)\frac{W}{\lambda_2+2\mu_2}\right]+\tfrac{1}{2}\frac{W}{\lambda_2+2\mu_2}[1+\rho_2-2\alpha_2(9\rho_2^{-2}+9\rho_2^{-1}+4+\rho_2)]\right\} \tag{24}$$

$$P^{(2)}_{\theta/\theta} = \tfrac{1}{8}\mu_1\mu_2\,\mathrm{e}^{-\rho_2}\frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2}\frac{1}{3\lambda_1+2\mu_1}\frac{V}{\Delta}\left\{\frac{3\lambda_2+2\mu_2}{\lambda_2+2\mu_2}\right.$$
$$\times\frac{W}{U}[3U(1+\rho_2)+(3\lambda_1+2\mu_1+4\mu_2)(3+3\rho_2+\rho_2^2)]$$
$$+[-4\mu_2\rho_2^{-2}+6\alpha_2(\lambda_2+2\mu_2)(3+3\rho_2+\rho_2^2)]\frac{W}{\lambda_2+2\mu_2}$$
$$\left.+\left[\frac{1}{U}(3\lambda_2+2\mu_2)(3\lambda_1+2\mu_1+4\mu_2)+2\mu_2-36\alpha_2\rho_2^{-2}(\lambda_2+2\mu_2)\right]\frac{1+\rho_2}{\lambda_1+2\mu_1}\right\}. \tag{25}$$

When l_1 and l_2 vanish the quantities $\rho_1 = r_0/l_1$ and $\rho_2 = r_0/l_2$ tend to infinity. In this case we obtain

$$\lim_{\rho_2\to\infty}(\lim_{\rho_1\to\infty}\sigma_\theta^{(2)}) = \frac{3\lambda_2(3\lambda_1+2\mu_1)+6\mu_2(3\lambda_2+2\mu_2)}{(3\lambda_2+2\mu_2)(3\lambda_1+2\mu_1+4\mu_2)}.$$

This value of $\sigma_\theta^{(2)}$ agrees with classical theory. In (21)–(25) we used the following abbreviated forms:

$$N = \frac{\sinh\rho_1-\rho_1\cosh\rho_1}{V}$$
$$U = 3\lambda_2+2\mu_2-3\lambda_1-2\mu_1$$
$$V = 3\sinh\rho_1-3\rho_1\cosh\rho_1+\rho_1^2\sinh\rho_1 \tag{26}$$
$$W = 4\alpha_2\rho_2^2\frac{\lambda_2+2\mu_2}{\lambda_1+2\mu_1}-N-4\alpha_1\rho_1^{-2}$$

and

$$\Delta = \tfrac{1}{8}\mu_1\mu_2\,\mathrm{e}^{-\rho_2}\frac{3\lambda_2+2\mu_2}{3\lambda_1+2\mu_1}\frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2}\frac{V}{U}\left\{[3U(1+\rho_2)\right.$$
$$+(3\lambda_1+2\mu_1+4\mu_2)(3+3\rho_2+\rho_2^2)]\frac{W}{\lambda_2+2\mu_2}$$
$$\left.+(3\lambda_1+2\mu_1+4\mu_2)\frac{1+\rho_2}{\lambda_1+2\mu_1}\right\}. \tag{27}$$

Note that whenever $3\lambda_2+2\mu_2 = 3\lambda_1+2\mu_1$ (i.e. $U = 0$) the double stresses vanish regardless of any discontinuity in the micromoduli.

(b) *Kinematical contact as in classical elasticity and partial transmissibility of energy*

It is possible to consider boundary conditions in which the strains are discontinuous at the interface. From the kinematical point of view such a contact resembles the classical notion of a bond. Even so, for a non planar boundary the solution is found to differ from that obtained in classical elasticity. Let the boundary conditions at $r = r_0$ be

$$u^{(1)} = u^{(2)} \qquad P_r^{(1)} = P_r^{(2)} \qquad R^{(1)} = 0 \qquad R^{(2)} = 0. \tag{28}$$

Employing (28) to solve for A, B, C and D in (18) and (19) and utilizing the principle of superposition, we obtain the following expressions at $r = r_0$

$$\sigma_\theta^{(1)} = \mu_2 \, e^{-\rho_2} \frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2} \frac{1}{\Delta^1} \left\{ \frac{\mu_2}{3\lambda_2+2\mu_2} [1+\rho_2+4\alpha_2\rho_2^{-2} \times(3+3\rho_2+\rho_2^2)] + 3\alpha_2\rho_2^{-2}(1+\rho_2) + \tfrac{1}{4}(1+\rho_2-4\alpha_2) \right\} \tag{29}$$

$$\sigma_\theta^{(2)} = \mu_2 \, e^{-\rho_2} \frac{1}{3\lambda_1+2\mu_1} \frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2} \frac{1}{\Delta^1} \Big\{ \mu_2[1+\rho_2+4\alpha_2\rho_2^{-2} \times(3+3\rho_2+\rho_2^2)] + 3\alpha_2\rho_2^{-2}(3\lambda_2+2\mu_2)(1+\rho_2) + \tfrac{1}{4}(3\lambda_1+2\mu_1) \times(1+\rho_2+4\alpha_2) + \frac{3\lambda_2+2\mu_2-3\lambda_1-2\mu_1}{3\lambda_2+2\mu_2}\left[\frac{\mu_2}{2}(1+\rho_2) + \alpha_2(3\lambda_2+2\mu_2)\right]\Big\} \tag{30}$$

$$\frac{1}{r_0}\bar{\mu}_{r\theta\theta}^{(2)} = \tfrac{1}{2}\mu_2\alpha_2(\lambda_2+\mu_2)\, e^{-\rho_2} \frac{1}{\Delta^1} \rho_2^{-2} \frac{3\lambda_2+2\mu_2-3\lambda_1-2\mu_1}{(3\lambda_2+2\mu_2)(3\lambda_1+2\mu_1)} \times(1+\rho_2)(5-4\alpha_2) \tag{31}$$

where

$$\Delta^1 = \mu_2 \, e^{-\rho_2} \frac{1}{3\lambda_1+2\mu_1} \frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2} \{\mu_2[1+\rho_2+4\alpha_2\rho_2^{-2}(3+3\rho_2+\rho_2^2)] + 3\alpha_2\rho_2^{-2}(3\lambda_2+2\mu_2)(1+\rho_2) + \tfrac{1}{4}(3\lambda_1+2\mu_1)(1+\rho_2+4\alpha_2)\} \tag{32}$$

In this case the coefficient B vanishes so that no hyperstresses develop within the inclusion. The hyperstresses vanish in the exterior when $\lambda_1+2\mu_1 = \lambda_2+2\mu_2$, regardless of the values of l_1, l_2, α_1 and α_2.

The results of this section do not depend on α_1 and ρ_1 and a reduction to classical results is effected by taking a single limit. We obtain

$$\lim_{\rho_2 \to \infty} \sigma_\theta^{(2)} = \sigma_\theta^{(2)\text{classical}}$$

(c) *Cavity*

For a cavity the boundary conditions at $r = r_0$ are

$$P_r^{(2)} = 0, \qquad R^{(2)} = 0. \tag{33}$$

The solution is given by (19) and yields the following values for the coefficients C and D:

$$C = \frac{1}{12\Delta_c} \frac{\mu_2}{\alpha_2} \frac{1}{\lambda_2+2\mu_2} e^{-\rho_2}\rho_2^2[1+\rho_2+4\alpha_2\rho_2^{-2}(3+3\rho_2+\rho_2^2)] \tag{34}$$

$$D = \frac{1}{\Delta_c} \cdot \frac{\mu_2}{2(\lambda_2 + 2\mu_2)} \tag{35}$$

where

$$\Delta_c = \frac{\mu_2}{\lambda_2 + 2\mu_2} e^{-\rho_2} \left[\frac{\mu_2}{6\alpha_2} \rho_2^2 (1+\rho_2) + \tfrac{2}{3}\mu_2 (3 + 3\rho_2 + \rho_2^2) + \tfrac{1}{2}(3\lambda_2 + 2\mu_2)(1+\rho_2) \right] \tag{36}$$

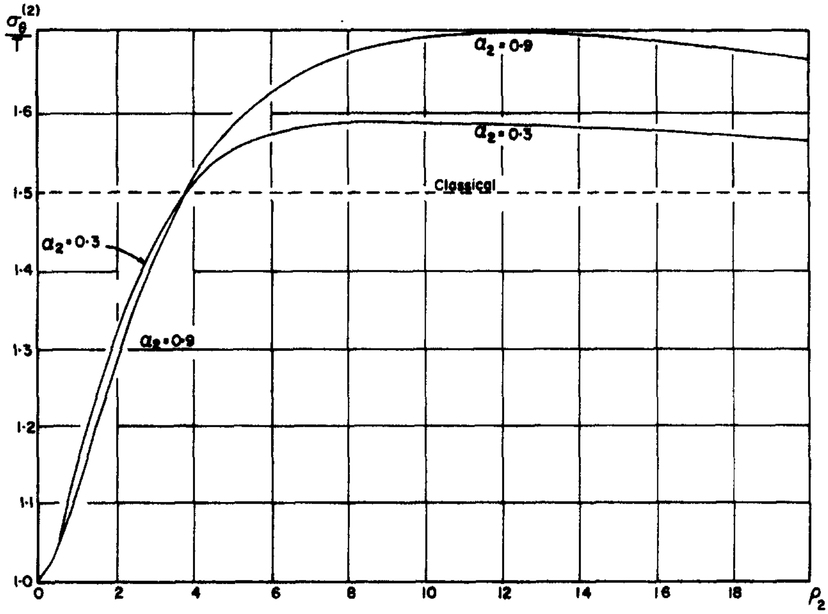

FIG. 3. Values of $\sigma_\theta^{(2)}/T$ at the surface $r = r_0$ of the cavity versus ρ_2 with $\lambda_2 = 2\mu_2$.

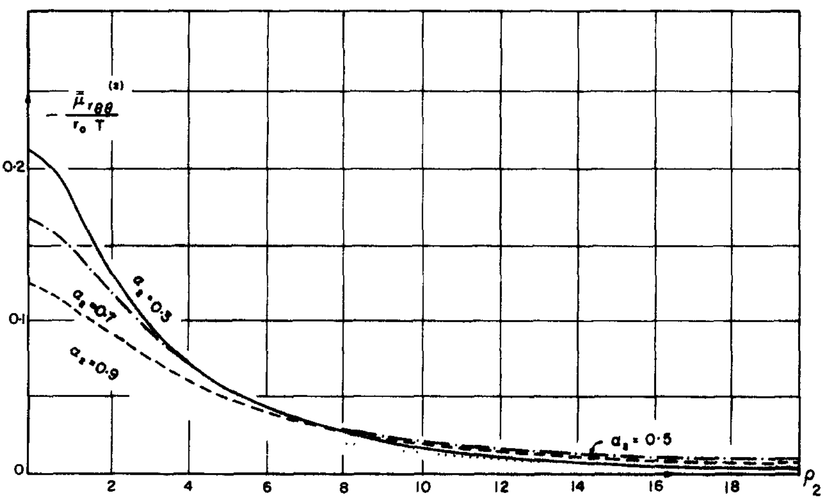

FIG. 4. Values of $-\bar{\mu}_{r\theta\theta}^{(2)}/r_0 T$ at the surface $r = r_0$ of the cavity versus ρ_2 with $\lambda_2 = 2\mu_2$.

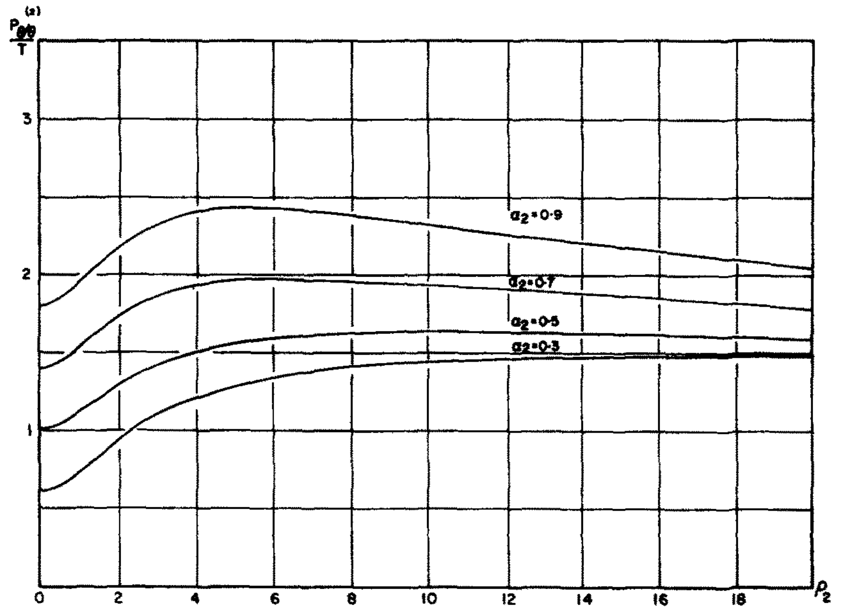

FIG. 5. Values of $P_{\theta\theta}^{(2)}/T$ at the surface $r = r_0$ of the cavity versus ρ_2 with $\lambda_2 = 2\mu_2$.

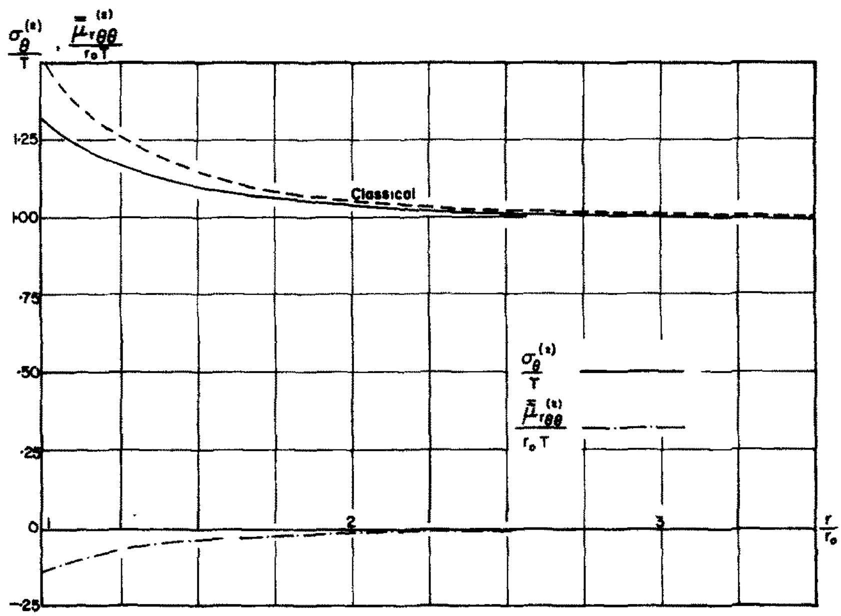

FIG. 6. Field values of $\sigma_\theta^{(2)}/T$, $\sigma_\theta^{(2)\text{classical}}/T$ and $\bar{\bar{\mu}}_{r\theta\theta}^{(2)}/r_0 T$ for a cavity versus r/r_0 with $\rho_2 = 2$, $\lambda_2 = 2\mu_2$.

At $r = r_0$ we obtain the following expressions:

$$\sigma_\theta^{(2)} = \frac{1}{\Delta_c}\frac{\mu_2}{\lambda_2+2\mu_2}e^{-\rho_2}\rho_2^2\left[\frac{1}{4}\frac{\mu_2}{\alpha_2}(1+\rho_2)+\tfrac{1}{2}(\lambda_2+2\mu_2)\rho_2^{-2}(3+3\rho_2+\rho_2^2)\right] \tag{37}$$

$$\frac{1}{r_0}\bar{\bar{\mu}}_{r\theta\theta}^{(2)} = -\frac{1}{12\Delta_c}\mu_2 e^{-\rho_2}(1+\rho_2)(5-4\alpha_2) \tag{38}$$

$$P_{\theta/\theta}^{(2)} = \frac{1}{\Delta_c}\frac{\mu_2}{\lambda_2+\mu_2}e^{-\rho_2}\rho_2^2\left[\frac{1}{4}\frac{\mu_2}{\alpha_2}(1+\rho_2)+\alpha_2\rho_2^{-2}(\lambda_2+2\mu_2)(3+3\rho_2+\rho_2^2)\right] \tag{39}$$

Upon taking the limit as $\rho_2 \to \infty$, we obtain the classical result $\sigma_\theta^{(2)} = \frac{3}{2}$.

The dependence of $\sigma_\theta^{(2)}$, $\bar{\bar{\mu}}_{r\theta\theta}^{(2)}$ and $P_{\theta/\theta}^{(2)}$ at $r = r_0$ upon ρ_2 and α_2 is shown in Figs. 3, 4 and 5.

In Fig. 3 $\sigma_\theta^{(2)}$ is the classical, short-range hoop-stress as influenced by strain-gradients. It is the same for all three forms of the theory.

In Fig. 4 $\bar{\bar{\mu}}_{r\theta\theta}^{(2)}$ represents one combination of long range forces (the self-equilibrating double-stresses as defined in form III) across a meridional section. Hyperstresses μ computed on the basis of the other two forms of the strain-gradient theory would represent a different combination of such forces and couple-stresses. Which of these combinations, if any, might be of importance, in a criterion of yielding or fracture, is not known.

The quantity $P_{\theta/\theta}$ depicted in Fig. 5 is σ_θ plus a certain combination of gradients of long range forces. The corresponding P from one of the other forms might, in general, represent σ_θ plus another combination of gradients of long range forces. Again, which of these, if any, might be of importance is not known.

The variation of $\sigma_\theta^{(2)}$ and $(1/r_0)\bar{\bar{\mu}}_{r\theta\theta}^{(2)}$ versus the dimensionless distance r/r_0 is compared with the classical case in Fig. 6.

CONCLUSIONS

The results indicate that the hyperstresses are largely confined to the interface $r = r_0$. Upon expanding the expressions for $\bar{\bar{\mu}}_{r\theta\theta}$ in power series one obtains the forms

$$\bar{\bar{\mu}}_{r\theta\theta}^{(1)} = a_1 r + a_2 r^2 + a_3 r^3 + \ldots \tag{40}$$

$$\bar{\bar{\mu}}_{r\theta\theta}^{(2)} = b_1 r^{-4} + (b_2 r^{-4} + b_3 r^{-3} + b_4 r^{-2} + \ldots)e^{-r} \tag{41}$$

where $a_1, a_2, \ldots b_1, b_2, \ldots$ are constants.

The quantity $\bar{\bar{\mu}}_{r\theta\theta}$ attains its largest value at $r = r_0$ and decreases very sharply as r increases. It is worth noting that in the present three dimensional case this decrease is characterized by a fourth power term (see equation (41)) in contrast to a third power term that dominates the decrease of $\bar{\bar{\mu}}_{r\theta\theta}$ in a two-dimensional case [4].

It is observed from Fig. 3 that the magnitude of the hoop-stresses at the surface of the cavity, which represent the stress concentration factor, differs substantially from the classical value. This difference remains significant even for large values of ρ_2. Moreover, the classical theory does not provide an upper bound for the stress concentration factor, and underestimates its magnitude for the physically feasible case of large ρ_2.

For a reduction to classical theory it is necessary to let $c_1, \bar{c}_1, c_2$, and $\bar{c}_2$ approach zero (such a reduction will be also achieved by letting ρ_1 and ρ_2 tend to infinity. In this case, however, c and $\bar{c}$ will approach zero simultaneously for each material). This process indeed effects a smooth reduction to the classical results for a cavity and an inclusion with kinematical contact as in classical theory. This reduction does not occur in a straight forward manner when a higher order contact prevails at the interface between the inclusion and the exterior region, because a contact of this nature demands continuity of the strain ε_r at $r = r_0$. The boundary conditions representing this non-classical bond are not likely to be completely eliminated when the micromoduli approach zero. In this case the limiting values of solutions to problems in strain-gradient theory should not be expected to agree with the classical results.

In the present problem the classical strains at the interface $r = r_0$ are

$$\begin{aligned} \varepsilon_r^{(1)\text{classical}} &= \frac{3(\lambda_2+2\mu_2)}{(3\lambda_2+2\mu_2)(3\lambda_1+2\mu_1+4\mu_2)} \\ \varepsilon_r^{(2)\text{classical}} &= \frac{3(3\lambda_1+2\mu_1-2\lambda_2)}{(3\lambda_2+2\mu_2)(3\lambda_1+2\mu_1+4\mu_2)} \end{aligned} \tag{42}$$

Their ratio is

$$\frac{\varepsilon_r^{(1)\text{classical}}}{\varepsilon_r^{(2)\text{classical}}} = \frac{\lambda_2+2\mu_2}{3\lambda_1+2\mu_1-2\lambda_2} \tag{43}$$

In strain-gradient theory with higher order contact at $r = r_0$ equation $(20)_2$ yields

$$\frac{\varepsilon_r^{(1)}}{\varepsilon_r^{(2)}} = 1$$

which contradicts (43).

When both ρ_1 and ρ_2 tend to infinity in the strain-gradient solution, the equality of the radial strains at $r = r_0$ is maintained

$$\varepsilon_r^{(1)} = \varepsilon_r^{(2)} = \varepsilon_r$$

but their *value* depends on the limiting process followed. We find

$$\lim_{\rho_2\to\infty} (\lim_{\rho_1\to\infty} \varepsilon_r) = \varepsilon_r^{(2)\text{classical}}$$

This limiting process leads to a classical exterior field, with classical $\sigma_\theta^{(2)}$ and $\varepsilon_r^{(2)}$, but yields non-classical values in the interior. By reversing the limiting process we obtain

$$\lim_{\rho_2\to\infty} (\lim_{\rho_1\to\infty} \varepsilon_r) = \varepsilon_r^{(1)\text{classical}}$$

and so recover the classical interior field. In this case we obtain non-classical values in the exterior.

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Résumé—Cette étude traite du domaine de la tension élastique due à une inclusion sphérique ou une cavité dans une région extérieure infinie soumise à une tension sphérique symétrique à l'infini. Les résultats sont dérivés pour une énergie potentielle qui dépend des tensions et des pentes de tension. Une telle energie potentielle fera naître des tensions classiques et des hypertensions non-classiques. Il est démontré que les hypertensions sont confinées aux vicinités de l'interface ou surface libre associée à une inclusion ou une cavité et sont en conséquent representatifs des effets de surface. Le facteur de concentration de tension, calculé sur la base de la tension en cercle autour d'une cavité, est prouvé être en grand désaccord avec la valeur classique

Zusammenfassung—Diese Abhandlung beschäftigt sich mit dem elastischen Spannungsfeld, veranlasst bei einem sphärischen Einschluss oder Aushölung in einem unendlichen äusseren Gebiet, welches einer sphärischen symmetrischen Spannung im Unendlichen unterworfen ist. Die Ergebnisse sind für eine potentielle Energie abgeleitet, welche von Beanspruchungen und Beanspruchungsgefällen abhängig ist. So eine potentielle Energie führt zu klassischen Beanspruchungen und zu nicht-klassischen Hyperbeanspruchungen. Es wird gezeigt dass die Hyperbeanspruchungen zu der Nähe der Grenzflächen der freien Oberfläche mit einem Einschluss oder einer Aushöhlung beschränkt sind und daher ein Grenzschicht Effekt sind. Es wird fernerhin gezeigt, dass der Konzentrations Beanspruchungsfaktor, berechnet auf der Grundlage der Randbeanspruchung ringsum einer Aushöhlung, von klassischen Wert sehr verschieden ist.

Абстракт—Эта статья рассматривает поле упругого напряжения, происходящее от сферического включения или от полости в бесконечном внешнем районе, подвергнутом сферически симметричному напряжению в бесконечности. Результаты выводятся для потенциальной энергии, которая зависит от напряжений и от градиентов напряжения. Такая потенциальная энергия даёт начало, как классичёским напряжениям, так и неклассическим гипернапряжениям. Указывается, что гипернапряжения ограничиваются близостью к поверхности раздела или свободной поверхности, связанной со вклюением или полостью и поэтому представляют из себя эффекты границмого слоя. Показатель концентрации напряжения, вычисленный на основании напряжений растяжений вокруг полости, оказывается в большом отклонении от классического значения величны.